JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 15, No. 2, March–April 1992

Near-Minimum-Time Control of a Flexible Manipulator

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Near-minimum-time control of flexible manipulators can be accomplished by designing a controller that tracks a reference maneuver. The control method presented here uses a near-minimum-time rigid-link reference maneuver to generate reference control torques and a Liapunov controller to make the flexible manipulator track the reference maneuver while reducing and eventually eliminating flexible motions. The near-minimum-time reference maneuver uses a smoothed bang-bang control plus a term to cancel the nonlinear dynamics of the rigid manipulator. The Liapunov function is a weighted sum of the energies of the elements of the flexible manipulator, and the control law is chosen to make the rate of change of the Liapunov function negative. The Liapunov function is bounded during a maneuver and decays asymptotically after the maneuver ends.

I. Introduction

PRECISE control of a manipulator is an important aspect of robot performance. In the past, this has led to the design of very stiff links for manipulators, which also leads to very heavy links, large motors, and slow response. To avoid these problems, lighter and more flexible manipulators will require advanced control strategies.

This work builds on results from the control of rigid robots[1] and the control of flexible structures. Fujii and Ishijima[2] developed a globally stable Liapunov controller to reorient a structure composed of a rigid hub and a flexible appendage. Junkins et al.[3] extended this control law to track a reference maneuver and showed that the Liapunov function is bounded during the maneuver and decays asymptotically to zero after the maneuver ends. Using a Gibbs phenomenon analysis, Baruh and Tadikonda[4] have shown that using smoothed control torque profiles as in Ref. 3 reduces the excitation of the flexible modes during a maneuver. Experimental work in this area is also being done.[5,6]

In this work, we demonstrate that the control design procedure of Ref. 3 can be developed, using work-energy relations, and we extend it to a multibody case, using a planar two-link manipulator to demonstrate the procedure. A schematic diagram of the manipulator is shown in Fig. 1. The control law developed for the two-link manipulator allows the flexible motions to be stabilized without any finite-dimensional estimation of the flexible modes, thus avoiding any spillover problems.

II. Linear Reference Problem

The dynamics of a rigid robot in the absence of gravity can be written in the form

$$M(\theta)\ddot{\theta} + N(\theta,\dot{\theta}) = u \tag{1}$$

where θ is the vector of joint angles, $M(\theta)$ is the mass matrix, and the vector $N(\theta,\dot{\theta})$ represents the nonlinear terms. If the mass matrix were not a function of the joint angles, the nonlinear terms would disappear, and the dynamics could be written as

$$\ddot{\eta} = u, \qquad \eta = M\theta \tag{2}$$

Minimum-time control of this system is very simple: $u_i = \pm u_{i\max}$, the sign chosen depending on the location of the system in the phase space (η_i, $\dot{\eta}_i$). Motivated by this simpler problem, we define a linear reference problem

$$\begin{gathered} \ddot{\eta} = u \\ \eta(t_o) = M[\theta(t_o)]\theta(t_o), \qquad \dot{\eta}(t_o) = 0 \\ \eta(t_f) = M[\theta(t_f)]\theta(t_f), \qquad \dot{\eta}(t_f) = 0 \\ |u_i| \le u_{i\max} \end{gathered} \tag{3}$$

To prevent discontinuities in the control from unduly exciting the flexible motions of the structure, u is defined using

$$u_i = \begin{cases} \pm u_{i\max}(3t^2/\delta^2 - 2t^3/\delta^3) & 0 \le t \le \delta \\ \pm u_{i\max} & \delta \le t \le t_1 \\ \pm u_{i\max}\left[1 - \dfrac{3(t-t_1)^2}{2\delta^2} + \dfrac{(t-t_1)^3}{2\delta^3}\right] & t_1 \le t \le t_2 \\ \mp u_{i\max} & t_2 \le t \le t_3 \\ \pm u_{i\max}\left[\dfrac{3(t-t_3)^2}{\delta^2} - \dfrac{2(t-t_3)^3}{\delta^3} - 1\right] & t_3 \le t \le t_f \end{cases} \tag{4}$$

where

$$\begin{aligned} t_1 &= t_f/2 - \delta \\ t_2 &= t_f/2 + \delta \\ t_3 &= t_f - \delta \end{aligned} \tag{5}$$

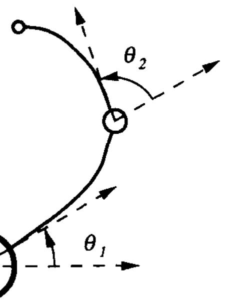

Fig. 1 Schematic diagram of a flexible, two-link robot arm.

Presented as Paper 90-2916 at the AIAA/AAS Astrodynamics Conference, Portland, OR, Aug. 20–22, 1990; received Nov. 2, 1990; revision received May 1, 1991; accepted for publication May 8, 1991.

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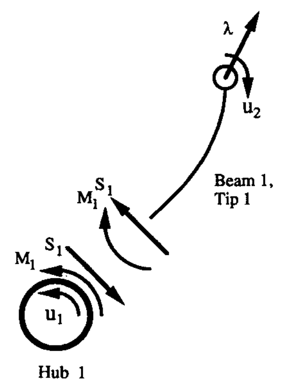

Fig. 2 Free-body diagram of the shoulder link.

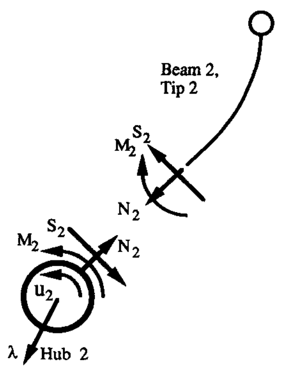

Fig. 3 Free-body diagram of the elbow link.

This is a near-minimum-time control. The parameter δ controls the smoothness of the control profile. Setting δ to 0 generates the bang-bang torque profile corresponding to minimum-time control of a single rigid body. As δ increases, the control profile becomes smoother at the expense of increased final time. Since both elements of η will not generally reach their final values simultaneously, one value of $u_{i\max}$ will be reduced so both have identical final times. This profile was introduced in Ref. 3, and Baruh and Tadikonda[4] have shown that this smooth approximation to a bang-bang control dramatically reduces the energy pumped into the flexible motions of a structure.

A computed torque controller to make a rigid robot follow the linear reference problem can be defined as

$$u_{\text{ref}} = u + K_p[\eta - M(\theta)\theta] + K_v[\dot{\eta} - M(\theta)\dot{\theta}] + N(\theta,\dot{\theta}) \qquad (6)$$

where K_p and K_v are constant gain matrices. Numerical solutions have verified that this control law yields excellent tracking of the reference inputs. Similar control laws have been demonstrated experimentally.[1]

III. Liapunov Controller

A control law for a flexible manipulator can be developed using the Liapunov control design method of Junkins et al.[3] Before designing the tracking control law, we will illustrate the procedure by developing a control law that drives the manipulator to a given set of joint angles while suppressing the flexible motions. Even though this control law is globally stable, it is inadequate for large angle maneuvers. Large gains that give good damping and precise position control near the desired position may require unacceptably large control torques for large angular displacements. This might exceed the actuator's capabilities or excite the flexible modes of the structure. These problems are especially acute when the controller is first turned on. To avoid these problems, the control law can be extended to track a reference maneuver, such as the one introduced in the previous section.

The Liapunov function for the nontracking controller is a weighted average of the structural energy of each element of the manipulator, plus extra terms that force the Liapunov function to be a minimum at the desired configuration of joint angles. The Liapunov function is

$$\begin{aligned} U &= a_1E_1 + a_2E_2 + a_3E_3 + a_4E_4 + \tfrac{1}{2}b_1(\theta_1 - \theta_{1\text{ref}})^2 \\ &\quad + \tfrac{1}{2}b_2[\theta_2 - r_1'(l_1) - \theta_{2\text{ref}}]^2 \end{aligned} \qquad (7)$$

where E_1 is the kinetic energy of the hub of the first link, E_2 is the kinetic energy plus the strain energy of the first beam and its tip mass, E_3 is the kinetic energy of the hub of the second link, E_4 is the kinetic energy plus the strain energy of the second beam and its tip mass and payload, and $\theta_{1\text{ref}}$ and $\theta_{2\text{ref}}$ are the desired reference joint angles. The quantity $r_1'(l_1)$ represents the angular deflection of the tip of the first link and is included because angle encoders measure the relative angle between the tip of one link and the base of the next. It can be verified that Eq. (10) is a positive definite function with its global minimum at the target state: $\theta_1 = \theta_{1\text{ref}}$, $\theta_2 = \theta_{2\text{ref}}$, $\dot{\theta}_1 = \dot{\theta}_2 = 0$, $y_1 = \dot{y}_1 = 0$, and $y_2 = \dot{y}_2 = 0$.

Since the elements of the arm are passive structural members, the rate of change of their energies is simply the applied external forces and moments times the velocities at the corresponding points of action. Referring to the free-body diagrams of Figs. 2 and 3, the rate of change of the Liapunov function in the absence of damping is

$$\begin{aligned} \dot{U} &= a_1\dot{\theta}_1[u_1 + (M_1 - r_1S_1)] - a_2\dot{\theta}_1(M_1 - r_1S_1) \\ &\quad + a_2\dot{r}_1(l_1)\lambda - a_2u_2[\dot{\theta}_1 + \dot{r}_1'(l_1)] - a_3\dot{r}_1(l_1)(\lambda + N_2 + S_2) \\ &\quad + a_3(\dot{\theta}_1 + \dot{\theta}_2)[u_2 + (M_2 - r_2S_2)] + a_4\dot{r}_1(l_1)(N_2 + S_2) \\ &\quad - a_4(\dot{\theta}_1 + \dot{\theta}_2)(M_2 - r_2S_2) + b_1\dot{\theta}_1(\theta_1 - \theta_{1\text{ref}}) \\ &\quad + b_2[\theta_2 - r_1'(l_1) - \theta_{2\text{ref}}][\dot{\theta}_2 - \dot{r}_1'(l_1)] \end{aligned} \qquad (8)$$

A control law that is a simple member of the previous family and uses easily measured quantities can be developed by choosing $a_2 = a_3 = a_4$. The $\dot{U}$ can be made negative by choosing the control torques to be

$$\begin{aligned} u_1 &= -g_{11}(\theta_1 - \theta_{1\text{ref}}) - g_{12}\dot{\theta}_1 - g_{13}(M_1 - r_1S_1) \\ u_2 &= -g_{21}[\theta_2 - r_1'(l_1) - \theta_{2\text{ref}}] - g_{22}[\dot{\theta}_2 - \dot{r}_1'(l_1)] \end{aligned} \qquad (9)$$

where $g_{11} = b_1/a_1$, $g_{13} = (a_1 - a_2)/a_1$, and $g_{21} = b_2/a_2$. The quantity $M_1 - r_1S_1$ can be measured using strain gauges at the base of link 1, whereas the angular positions and velocities can be measured using angle encoders and tachometers. This control law is similar to those used in Refs. 2 and 6. Using these controls, the rate of change of the Liapunov function becomes

$$\dot{U} = -g_{12}\dot{\theta}_1^2 - g_{22}[\dot{\theta}_2 - \dot{r}_1'(l_1)]^2 \qquad (10)$$

which is negative for all nonzero joint velocities. While this expression is only negative semidefinite, the closed-loop system is in equilibrium only when $U = 0$, so eventually the Liapunov function must go to zero. This is equivalent to saying that the structure's flexible and rigid-body motions are controllable. Note that g_{11}, g_{12}, g_{21}, and g_{22} must be greater than zero, whereas g_{13} must be greater than or equal to -1, to guarantee stability.

To extend this controller to track a reference motion, the Liapunov function is redefined in terms of energy errors, ΔE_i, where

$$\Delta E_1 = \tfrac{1}{2} I_{\text{hub }1}(\dot\theta_1 - \dot\theta_{1r})^2$$

$$\Delta E_2 = \tfrac{1}{2}\int_{r_1}^{l_1} \rho_1 \left|\dot y_{1(x_1,t)} + x_1\dot\theta_1 \hat j_1 - \dot y_{1r(x_1,t)} - x_1 \dot\theta_{1r}\hat j_1\right|^2 \mathrm{d}x_1 + \int_{r_1}^{l_1} f_1(y_1 - y_{1r}, y_1' - y_{1r}', y_1'' - y_{1r}'')\,\mathrm{d}x_1$$

$$\Delta E_3 = \tfrac{1}{2} I_{\text{hub }2}(\dot\theta_1 - \dot\theta_{1r} + \dot\theta_2 - \dot\theta_{2r})^2$$

$$\Delta E_4 = \tfrac{1}{2}\int_{r_2}^{l_2} \rho_2 \left|\dot y_1(l_1) + l_1\dot\theta_1\hat j_1 - \dot y_{1r}(l_1) - l_1\dot\theta_{1r}\hat j_1 + \dot y_{2(x_2)} + x_2(\dot\theta_1 + \dot\theta_2)\hat j_2 - \dot y_{2r(x_2)} - x_2(\dot\theta_{1r} + \dot\theta_{2r})\hat j_2\right|^2 \mathrm{d}x_2 + \int_{r_2}^{l_2} f_2(y_2 - y_{2r}, y_2' - y_{2r}', y_2'' - y_{2r}'')\,\mathrm{d}x_2 \tag{11}$$

where y_i represents the vector displacement of an element of the ith link from its original position, f_i is the strain energy per unit length of the ith link, and $(\,)_r$ quantities represent the open-loop response of the structure to the reference inputs. Note that we have made no kinematic assumptions about the deflections: geometric stiffening is allowed, small angles are not assumed, and even axial vibrations are possible. Although the generality of this formulation is inappropriate for calculations of mode shapes and similar quantities, it allows us to focus on basic physical quantities of the system, such as angular momentum and energy.

Deriving $\dot U$ for the modified Liapunov function involves deriving the equations of motion using Hamilton's principle, using these equations of motion to eliminate several terms, and then noting that the remaining integrals are the rates of change of linear and angular momentum. This leads to

$$\begin{aligned}\dot U = {}& (\dot\theta_1 - \dot\theta_{1r})\{a_1(u_1 - u_{1\text{ref}}) + b_1(\theta_1 - \theta_{1r}) + (a_1 - a_2)[(M_1 - r_1S_1) - (M_1 - r_1S_1)_r]\} \\ &+ [\dot\theta_2 - \dot r_1'(l_1) - \dot\theta_{2r} + \dot r_{1r}'(l_1)]\{a_2(u_2 - u_{2\text{ref}}) + b_2[\theta_2 - r_1'(l_1) - \theta_{2r} + r_{1r}'(l_1)]\}\end{aligned} \tag{12}$$

Rather than basing the control law on the open-loop response to the reference torques, we choose to base the control law on the rigid-link reference maneuver:

$$\begin{aligned} u_1 &= u_{1\text{ref}} + g_{11}(\theta_{1\text{ref}} - \theta_1) + g_{12}(\dot\theta_{1\text{ref}} - \dot\theta_1) + g_{13}[(M_1 - r_1S_1)_{\text{ref}} - (M_1 - r_1S_1)] \\ u_2 &= u_{2\text{ref}} + g_{21}\{\theta_{2\text{ref}} - [\theta_2 - r_1'(l_1)]\} + g_{22}\{\dot\theta_{2\text{ref}} - [\dot\theta_2 - \dot r_1'(l_1)]\}\end{aligned} \tag{13}$$

where $(\,)_{\text{ref}}$ quantities are based on the rigid-link reference maneuver of Sec. II, and g_{ij} are defined as before. After the maneuver terminates, $\dot\theta_{1\text{ref}}$, $(M_1 - r_1S_1)_{\text{ref}}$, and $\dot\theta_{2\text{ref}}$ become zero, and this control becomes the same as the one in Eq. (12). This gives a smooth transition to a control law that is asymptotically stable. During the maneuver,

$$\begin{aligned}\dot U = {}& -g_{12}(\dot\theta_1 - \dot\theta_{1r})^2 - g_{22}[\dot\theta_2 - \dot r_1'(l_1) - \dot\theta_{2r}]^2 \\ &+ (\dot\theta_1 - \dot\theta_{1r})\{b_1(\theta_{1\text{ref}} - \theta_{1r}) + a_1g_{12}(\dot\theta_{1\text{ref}} - \dot\theta_{1r}) + (a_1 - a_2)[(M_1 - r_1S_1)_{\text{ref}} - (M_1 - r_1S_1)_r]\} \\ &+ [\dot\theta_2 - \dot r_1'(l_1) - \dot\theta_{2r}]\{b_2[\theta_{2\text{ref}} - \theta_{2r} + r_{1r}'(l_1)] + a_2g_{22}[\dot\theta_{2\text{ref}} - \dot\theta_{2r} + \dot r_{1r}'(l_1)]\}\end{aligned} \tag{14}$$

Unfortunately, this is not negative semidefinite as before. The $\dot U$ will be negative whenever

$$\begin{aligned} g_{12}|\dot\theta_1 - \dot\theta_{1r}| &> |b_1(\theta_{1\text{ref}} - \theta_{1r}) + a_1g_{12}(\dot\theta_{1\text{ref}} - \dot\theta_{1r}) + (a_1 - a_2)[(M_1 - r_1S_1)_{\text{ref}} - (M_1 - r_1S_1)_r]| \\ g_{22}|\dot\theta_2 - \dot r_1'(l_1) - \dot\theta_{2r}| &> |b_2[\theta_{2\text{ref}} - \theta_{2r} + r_{1r}'(l_1)] + a_2g_{22}[\dot\theta_{2\text{ref}} - \dot\theta_{2r} + \dot r_{1r}'(l_1)]|\end{aligned} \tag{15}$$

The right-hand sides of these inequalities can be computed in advance of a given maneuver. These can be thought of as forcing terms that may drive the Liapunov function to a greater value, while the squared terms in Eq. 14 will prevent it from growing without bound. Simulations of the open- and closed-loop dynamics can determine the values of these terms and the value of the Liapunov function during the maneuver, to assess the performance of the control law using different values of the gains and the smoothing parameter δ.

IV. Discretized Equations of Motion

Simulating the motion of a flexible arm on a digital computer requires that the dynamics be formulated as a set of ordinary differential equations. The equations of motion for a two-link manipulator have been discretized using the assumed modes method. The Lagrangian is defined in terms of the rigid-body coordinates θ_1 and θ_2 and the amplitudes of the first link's modes a_i and the second link's modes b_i. Then Lagrange's equations yield the set of ordinary differential equations that govern the motion.

The kinetic energy in the modal formulation is[7]

$$\begin{aligned} T = {}& \tfrac{1}{2}(I_{1\text{rigid}} + m_2 l_1^2)\dot\theta_1^2 + \dot\theta_1 \dot a_i f_{1i} + \tfrac{1}{2}\dot a_i \dot a_j m_{1ij} + [\dot a_i \phi_i(l_1) + l_1\dot\theta_1]\dot b_j \cos\theta_2 e_{2j} + I_{2\text{rigid}}(\dot\theta_1 + \dot\theta_2)^2 \\ &+ (\dot\theta_1 + \dot\theta_2)\dot b_i f_{2i} + [\dot a_i\phi_i(l_1) + l_1\dot\theta_1](\dot\theta_1 + \dot\theta_2)\cos\theta_2 e_{20} + \tfrac{1}{2} m_{2ij}\dot b_i \dot b_j \end{aligned} \tag{16}$$

where $I_{1\text{rigid}}$ is the total rigid-body inertia of link one, m_2 is the total mass of link 2, $I_{2\text{rigid}}$ is the total rigid-body inertia of the second link plus the payload, $m_{1\text{tip}}$, $m_{2\text{tip}}$, $I_{1\text{tip}}$, and $I_{2\text{tip}}$ are the masses and moments of the tips of the links, and the mass moments are

$$\begin{aligned} m_{1ij} &= \int_{r_1}^{l_1} \rho_1\phi_i\phi_j\,\mathrm{d}x_1 + (m_{1\text{tip}} + m_2)\phi_i(l_1)\phi_j(l_1) + I_{1\text{tip}}\phi_i'(l_1)\phi_j'(l_1) \\ f_{1i} &= \int_{r_1}^{l_1} \rho_1\phi_i x_1\,\mathrm{d}x_1 + (m_{1\text{tip}} + m_2)l_1\phi_i(l_1) + I_{1\text{tip}}\phi_i'(l_1) \\ m_{2ij} &= \int_{r_2}^{l_2} \rho_2\psi_i\psi_j\,\mathrm{d}x_2 + m_{2\text{tip}}\psi_i(l_2)\psi_j(l_2) + I_{2\text{tip}}\psi_i'(l_s)\psi_j'(l_s) \\ f_{2i} &= \int_{r_2}^{l_2} \rho_2\psi_i x_2\,\mathrm{d}x_2 + m_{2\text{tip}}l_2\psi_i(l_2) + I_{2\text{tip}}\psi_i'(l_2) \\ e_{20} &= \int_{r_2}^{l_2} \rho_2 x_2\,\mathrm{d}x_2 + m_{2\text{tip}}l_2 \\ e_{2i} &= \int_{r_2}^{l_2} \rho_2\psi_i\,\mathrm{d}x_2 + m_{2\text{tip}}\psi_i(l_2) \end{aligned} \tag{17}$$

where ϕ_i = the ith shape function of the first link, and ψ_i = the ith shape function of the second link. The potential energy in the modal formulation is

$$V = \tfrac{1}{2} a_i a_j k_{1ij} + \tfrac{1}{2} b_i b_j k_{2ij} \tag{18}$$

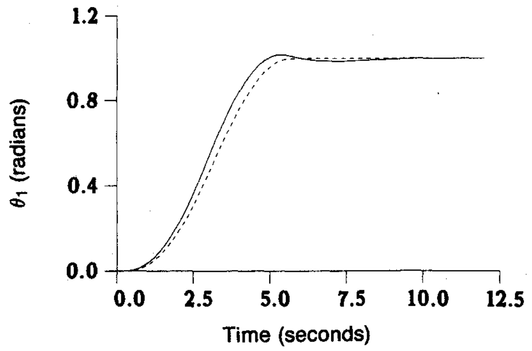

Fig. 4 Shoulder joint angle vs time; solid line: flexible motion of the arm, dashed line: rigid-body reference motion.

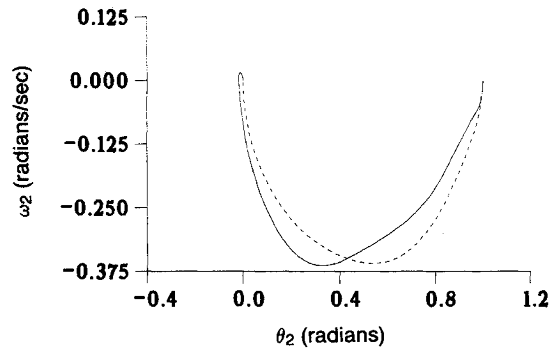

Fig. 7 Elbow joint phase portrait; solid line: flexible motion of the arm, dashed line: rigid-body reference motion.

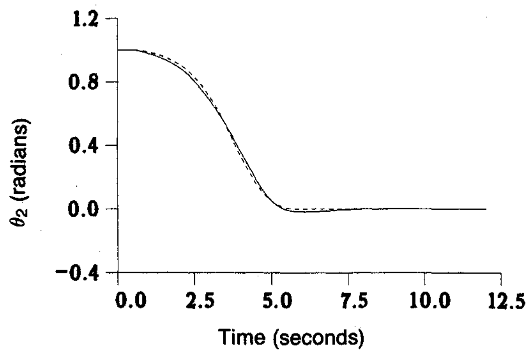

Fig. 5 Elbow joint angle vs time; solid line: flexible motion of the arm, dashed line: rigid-body reference motion.

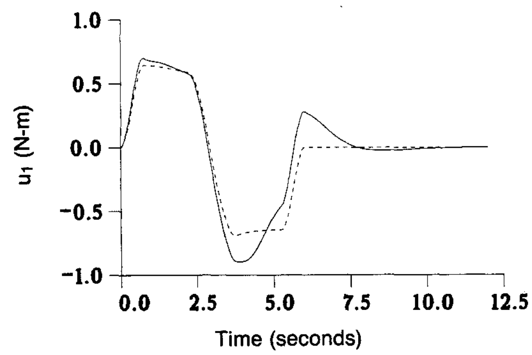

Fig. 8 Shoulder joint control torques; solid line: feedback control for the flexible arm, dashed line: open-loop reference control.

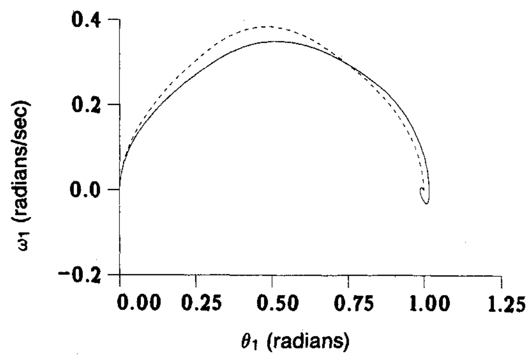

Fig. 6 Shoulder joint phase portrait; solid line: flexible motion of the arm, dashed line: rigid-body reference motion.

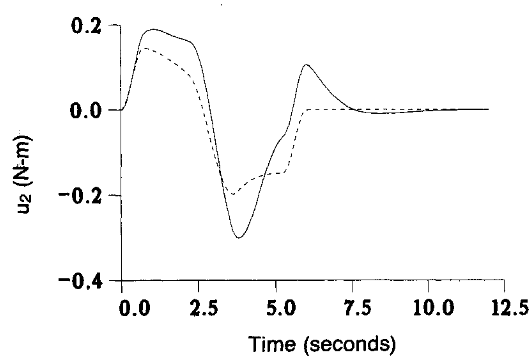

Fig. 9 Elbow joint control torques; solid line: feedback control for the flexible arm, dashed line: open-loop reference control.

where the stiffness moments are

$$k_{1ij} = \int_{r_1}^{l_1} EI_1 \phi_i'' \phi_j'' \, dx_1$$

$$k_{2ij} = \int_{r_2}^{l_2} EI_2 \psi_i'' \psi_j'' \, dx_2 \tag{19}$$

The external work done by the control torques on the system is

$$\delta W = u_1 \delta\theta_1 + u_2[\delta\theta_2 - \phi_i'(l_1)\delta a_i] \tag{20}$$

The shape functions adopted for both links were the eigenfunctions for a clamped-free beam with finite tip masses and rotational inertias. To avoid convergence difficulties, we found it useful to calculate the shape functions for the first beam using the undeformed inertial parameters of the second link as the tip parameters, whereas the calculations of m_{1ij} and f_{1i} were made using the actual tip mass parameters. Since eigenfunctions were used, $m_{2ij} = \delta ij$, and k_{1ij} and k_{2ij} were diagonal. The dynamical equations resulting from these energy approximations are omitted here for brevity's sake.

The control torques were based on the combinations of state variables that corresponded to the measurements needed. The implementation of the control law was

$$u_1 = u_{1\mathrm{ref}} + g_{11}(\theta_{1\mathrm{ref}} - \theta_1) + g_{12}(\dot\theta_{1\mathrm{ref}} - \dot\theta_1)$$
$$+ g_{13}\{(M_1 - r_1 S_1)_{\mathrm{ref}} - EI_1[\phi_i''(r_1) - r_1\phi_i'''(r_1)]a_i\}$$
$$u_2 = u_{2\mathrm{ref}} + g_{21}\{\theta_{2\mathrm{ref}} - [\theta_2 - \phi_i'(l_1)a_i]\}$$
$$+ g_{22}\{\dot\theta_{2\mathrm{ref}} - [\dot\theta_2 - \phi_i'(l_1)\dot a_i]\} \tag{21}$$

V. Results

The equations of motion derived using the assumed modes method have been implemented in a digital computer simulation. Figures 4–9 show results from the simulation. The results were based on using the first two flexible modes for each link. No noise was introduced into the measurements or controls. The control gains were $g_{11} = 800$ oz-in./rad, $g_{12} = 1500$ oz-in./(rad/s), $g_{13} = -0.25$ oz-in./(oz-in.), $g_{21} = 800$ oz-in./rad, and $g_{22} = 200$ oz-in./(rad/s). These gains were chosen to give "good" response without greatly exceeding the nominal control bounds; we did not attempt to formally optimize the gains.

Plots of the joint angles and joint phase portraits are given in Figs. 4–7, with the solid lines showing the flexible motion and the dotted lines showing the rigid reference maneuver. These show good tracking of the reference trajectory, with a slight overshoot of the joint angles as the maneuver ends. The control torques do show noticeable departures from the rigid-body reference torques as they compensate for the elastic motions of the links, as Figs. 8 and 9 show.

The reference maneuver for these plots was designed using $\ddot{\eta}_{\max} = (1.27,\ 0.149)^T$ N-m $= (180,\ 21)^T$ oz-in., which was shaped using the torque profile of Eq. 4, along with a nonlinearity canceling term, as in Eq. 6. The magnitude of the shoulder control torque was reduced to match the final times. The reference control torques (dashed lines) and the actual control torques (solid lines) are presented in Figs. 8 and 9. These figures show that the nonlinearity canceling term, which is proportional to $\dot{\theta}_2$, has biased the reference torques in a negative direction. The actual shoulder torques were close to the reference torques, whereas the actual elbow torques were larger than their reference values.

This demonstrates that this reference maneuver strategy does not take full advantage of the capabilities of the actuators. A superior strategy would use an optimization code to determine when the control torques should change signs, so the control torques could remain at their maximum levels through a greater portion of the trajectory.

The tracking control law successfully keeps the flexible arm near the rigid arm reference motion. The amount of moment feedback at the shoulder did not appear to have much effect on the overall trajectory, as other runs have shown. However, experimental work has demonstrated the usefulness of this kind of feedback, so further study of this is recommended.

VI. Conclusion

Using a Liapunov control design method, we have designed a stable tracking control law for a flexible robot arm. It is an extension of the standard proportional-derivative (PD) controller of classical control theory. In addition to feedback of the joint angles and joint velocities, the controller uses feedback of the shear and bending moment reactions at the roots of the flexible beams (which can be measured by strain gauges) to give the analyst another free parameter when designing the controller.

The derivation of this control law does not depend on any discretization scheme or any particular formulation of the dynamics of the flexible beams. Instead, it uses the general concepts of energy and angular momentum to prove that the control law leads to bounded errors while tracking an approximate reference maneuver, with the errors decaying asymptotically after the maneuvers end. Explicit feedback gain inequalities establish stability of the closed-loop endgame. The use of feedback of the shear and bending moment reactions at the roots of the flexible beams allows the controller to suppress structural vibrations without recourse to a finite-dimensional state estimator that may become destabilized by unmodeled dynamics.

The reference maneuver presented here uses a rigid-body trajectory that is driven by a smoothed bang-bang control profile plus a term to cancel modeled nonlinearities. This provides an analytically tractable reference motion, although more sophisticated methods deserve attention. In particular, determining true minimum-time reference maneuvers based on optimizing switch times deserves attention.

The tracking control law has been implemented in a digital computer simulation of a flexible, planar, two-link robot arm. Based on our results, we believe that establishing the practical usefulness of root moment feedback at the joints requires further study.

Besides elastic motion, the sources of inaccuracy for robots include backlash, compliance, and other nonlinearities. Although these were not considered in this study, they deserve consideration. Experimental work in particular would be useful in evaluating these issues.

Acknowledgments

The support of the Air Force Office of Scientific Research, Contract F49620-87-C-0078, and A. K. Amos is appreciated. We also received support from the Texas Advanced Technology Program, Project 70110. The authors wish to thank S. R. Vadali for his suggestion of the transformation in Eq. (2).

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